The nonlinear evolution of ring dark solitons in Bose–Einstein condensates

2004 J. Phys. A: Math. Gen. 37 11223

(http://iopscience.iop.org/0305-4470/37/46/008)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **37** (2004) 11223–11228 PII: S0305-4470(04)82810-5

The nonlinear evolution of ring dark solitons in Bose–Einstein condensates

Ju-Kui Xue

Physics and Electronics Engineering College, Northwest Normal University, Lanzhou, 730070, People's Republic of China

E-mail: Xuejk@nwnu.edu.cn

Received 29 June 2004, in final form 24 September 2004
Published 3 November 2004
Online at stacks.iop.org/JPhysA/37/11223
doi:10.1088/0305-4470/37/46/008

Abstract

The dynamics of the ring dark soliton in a Bose–Einstein condensate (BEC) with thin disc-shaped potential is investigated analytically and numerically. Analytical investigation shows that the ring dark soliton in the radial non-symmetric cylindrical BEC is governed by a cylindrical Kadomtsev–Petviashvili equation, while the ring dark soliton in the radial symmetric cylindrical BEC is governed by a cylindrical Korteweg–de Vries equation. The reduction to the cylindrical KP or KdV equation may be useful to understand the dynamics of a ring dark soliton. The numerical results show that the evolution properties and the snaking of a ring dark soliton are modified significantly by the trapping.

PACS numbers: 03.75.Fi, 03.65.Ge

1. Introduction

The ring dark soliton was first introduced in optics by Kivshar and Yang in [1] and was studied theoretically [2–4] and experimentally [5, 6]. The introduction of the ring dark soliton is based on the evolution properties of the dark stripe solitons. It is well known that, in two dimensions, the dark stripe solitons can be formed and become unstable against transverse snaking [7]. However, the instability band of the dark stripe solitons, characterized by a maximum perturbation wavenumber $Q_{\max}$, may be suppressed by bending a dark stripe to close it into an annulus of length $L < 2\pi/Q_{\max}$ [1]. On the other hand, the soliton is expected to have a circular symmetry and the stable ring dark soliton can be observed. In a recent work [8], the concept of a ring dark soliton in Bose–Einstein condensates (BEC) is first introduced and the ring dark soliton in a BEC with disc-shaped trap is studied, and predicts the existence of both oscillatory and stationary ring dark solitons. The numerical results show that instabilities gradually set in and, as a result, a shallow ring dark soliton slowly decays, while the deeper ones develop the snake instability, splitting into ring shaped vortex arrays. The dominant source of the instability is due to the fact that the ring shaped soliton carries a definite nonzero-angular momentum, which results in the growth of azimuthal perturbations. However, the dynamics of the nonlinear evolution of ring dark soliton remains an open question, especially in BEC. It is the aim of this paper to discuss the dynamics of a ring dark soliton in a BEC with disc-shaped trap. By using a perturbation method, we show that the ring dark soliton in the radial non-symmetric cylindrical BEC (i.e., with azimuthal effect) is governed by a cylindrical Kadomtsev–Petviashvili (cylindrical KP) equation, while the ring dark soliton in the radial symmetric cylindrical BEC (i.e., without azimuthal effect) is governed by a cylindrical Korteweg–de Vries (cylindrical KdV) equation.

2. The governing equations

The evolution of the weakly coupled BEC at low temperature is governed by the time-dependent Gross–Pitaevskii (GP) equation with the external disc-shaped potential $V(r)$

$$\mathrm{i}\frac{\partial\Psi}{\partial t} = \left[-\frac{1}{2}\nabla^2 - \frac{1}{2}\frac{\partial^2}{\partial z^2} + V(r) + Q|\Psi|^2\right]\Psi, \tag{1}$$

and the disc-shaped trap has the form

$$V(r) = V_r + V_z = \frac{1}{2}\left(\frac{\omega_r}{\omega_z}\right)^2 r^2 + \frac{1}{2}z^2, \tag{2}$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, $r^2 = x^2 + y^2$. ω_r and ω_z are frequencies of the trap in the radial r and axial z direction, respectively. Wavefunction Ψ, time t and variables (r, z) are normalized by $\left(N/a_z^3\right)^{1/2}$, ω_z^{-1} and a_z respectively, where $a_z = [\hbar/(m\omega_z)]^{1/2}$, $Q = 4\pi N a_s/a_z$ with a_s the s-wave scattering length, m the mass of the atom and N the number of atoms in BEC.

In order to investigate the dynamics of the ring solitary wave in BEC, we consider the solitary wave travelling along the radial direction and with transverse effect, i.e., $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{1}{r}\frac{\partial}{\partial r}$. Here we consider the excitation created in the BEC with very thin disc-shaped trap, i.e., the case in which the trapping potential in the r direction is much weaker than that in the z direction, mathematically, $\omega_r/\omega_z \ll 1$. This means that the motion of atoms in the z direction is essentially frozen and is governed by the ground-state wavefunction of the corresponding harmonic oscillator. Hence, the excitations can propagate only in the (r, θ) plane. According to the above assumption, we can set [9]

$$\Psi(r, \theta, z, t) = G_0(z)\Phi(r, \theta, t), \tag{3}$$

where $G_0(z) = \exp(-z^2/2)$ is the ground-state wavefunction of the 1D harmonic oscillator with the potential $z^2/2$ in the z direction. Then, substituting equation (3) into equation (1), we obtain

$$\mathrm{i}\frac{\partial\Phi}{\partial t} = \left[-\frac{1}{2}\nabla^2 + V(r) + Q'|\Phi|^2\right]\Phi, \tag{4}$$

where $Q' = I_0 Q$ is an effective interaction constant with $I_0 = \int_{-\infty}^{\infty}\mathrm{d}z\, G_0^4(z)\Big/\int_{-\infty}^{\infty}\mathrm{d}z\, G_0^2(z) = 1/\sqrt{2}$. Because the contribution of the higher-order eigenmodes of the harmonic oscillator in the z direction is very small and can be safely neglected, so, in deducing equation (4), we have multiplied equation (1) by $G_0^*(z)$ and then integrated once with respect to z to eliminate the dependence on z [10, 9]. Now we seek for a solution to equation (4) in the form [9]

$$\Phi(r, z, t) = A(r, \theta, t)\exp[-\mathrm{i}\mu t + \mathrm{i}\phi(r, \theta, t)], \tag{5}$$

where μ is the chemical potential of the condensate and ϕ is a phase function contributed from the excitation. Then, equation (4) reduces to

$$\frac{\partial A}{\partial t}+\frac{\partial A}{\partial r}\frac{\partial \phi}{\partial r}+\frac{1}{r^2}\frac{\partial A}{\partial \theta}\frac{\partial \phi}{\partial \theta}+\frac{A}{2}\left(\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}\right)=0, \tag{6}$$

$$-\frac{1}{2}\left(\frac{\partial^2 A}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 A}{\partial\theta^2}+\frac{1}{r}\frac{\partial A}{\partial r}\right)-\left(\mu-\frac{1}{2}\right)A$$
$$+\left[\frac{\partial\phi}{\partial t}+V_r+\frac{1}{2}\left(\frac{\partial\phi}{\partial r}\right)^2+\frac{1}{2}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)^2\right]A+Q'A^3=0. \tag{7}$$

3. The dynamics of the ring dark soliton and discussions

To obtain the nonlinear evolution of the ring dark soliton, we consider the excitation with nearly cylindrical symmetry, the angle subtended by the wave front in the nearly concentric case is small. Hence, the independent variables can be stretched as [11, 12] $\xi=\epsilon(r-ct)$, $\eta=\epsilon^{-1}\theta$ and $t=\epsilon^3\tau$, and the dependent variables can be scaled as

$$A=u_0+\epsilon^2(a_0+\epsilon^2 a_1+\cdots), \tag{8}$$

$$\phi=\epsilon(\phi_0+\epsilon^2\phi_1+\cdots), \tag{9}$$

where ϵ is a smallness ordering parameter characterizing the relative amplitude of the excitation. When $\epsilon\to 0$, equations (8) and (9) indicate that the condensate background, i.e., without perturbation, is recovered.

Then, substituting the above expansions into equations (6) and (7) and collecting the terms in the different powers of ϵ, we can obtain each nth-order reduced equation. In order to get insight into the ring dark soliton analytically, we can neglect the slowly varying radial trapping potential because ω_r/ω_z is very small (this term will be included in the numerical simulations). Hence, to the leading order, we have

$$-c\frac{\partial a_0}{\partial\xi}+\frac{1}{2}u_0\frac{\partial^2\phi_0}{\partial\xi^2}=0, \tag{10}$$

$$-2Q'u_0^2a_0+cu_0\frac{\partial\phi_0}{\partial\xi}=0, \tag{11}$$

which results in

$$a_0=\frac{u_0}{2c}\frac{\partial\phi_0}{\partial\xi}, \tag{12}$$

and $c^2=Q'u_0^2=\mu-1/2$ is also obtained from the solvability condition. At the next order, with the consideration of equation (12), we have

$$-c\frac{\partial a_1}{\partial\xi}+\frac{1}{2}u_0\frac{\partial^2\phi_1}{\partial\xi^2}=-\frac{\partial a_0}{\partial\tau}-\frac{3c}{u_0}a_0\frac{\partial a_0}{\partial\xi}-\frac{c}{\tau}a_0+\frac{u_0}{2c^2}\frac{1}{\tau^2}\frac{\partial^2\phi_0}{\partial\eta^2}, \tag{13}$$

$$-2c^2a_1+cu_0\frac{\partial\phi_1}{\partial\xi}=2c\int\frac{\partial a_0}{\partial\tau}\,\mathrm{d}\xi-\frac{1}{2}\frac{\partial^2 a_0}{\partial\xi^2}+\frac{3c^2}{u_0}a_0^2. \tag{14}$$

The solvability condition for equations (13) and (14) reads

$$\frac{\partial}{\partial\xi}\left[\frac{\partial a_0}{\partial\tau}+\alpha a_0\frac{\partial a_0}{\partial\xi}-\beta\frac{\partial^3 a_0}{\partial\xi^3}+\frac{1}{2\tau}a_0\right]+\frac{1}{2c}\frac{1}{\tau^2}\frac{\partial^2 a_0}{\partial\eta^2}=0, \tag{15}$$

where $\alpha = 3c/u_0$, $\beta = 1/(8c)$. Equation (15) is the cylindrical KP equation [11] describing the small-amplitude ring dark soliton in the radial non-symmetric cylindrical BEC. The last term on the left-hand side of equation (15) refers to the transverse effect.

If the wave propagates in a radial symmetric cylindrical BEC, the last term on the left-hand side of equation (15) disappears and the cylindrical KP equation (15) reduces to the cylindrical KdV equation

$$\frac{\partial a_0}{\partial \tau} + \alpha a_0 \frac{\partial a_0}{\partial \xi} - \beta \frac{\partial^3 a_0}{\partial \xi^3} + \frac{1}{2\tau} a_0 = 0. \tag{16}$$

This cylindrical KdV equation describes the dynamics of the ring dark soliton in BEC without the azimuthal effect. The ring dark quasi-solitary wave solution of equation (16) for larger radius is [13]

$$a_0 = -A \left(\frac{\tau_0}{\tau}\right)^{2/3} \operatorname{sech}^2 \left\{ \left(\frac{\alpha A}{12\beta}\right)^{1/2} \left(\frac{\tau_0}{\tau}\right)^{1/3} \left[\xi + \frac{\alpha A}{3} \left(\frac{\tau_0}{\tau}\right)^{2/3} \tau \right] \right\}, \tag{17}$$

where A is the amplitude of the ring solitary wave at initial stage τ_0. It is clear that equation (17) describes a ring dark soliton and its amplitude varies according to $\tau^{-2/3}$ law. That is, the soliton shape, amplitude and phase are varying when the soliton propagates in BEC, which is confirmed by the experiments as mentioned before. In this case, however, the ring shape is not deformed as the ring dark soliton propagates.

Equation (15) also indicates that for sufficiently large τ, i.e., for a very broad ring dark soliton, we have

$$\frac{\partial a_0}{\partial \tau} + \alpha a_0 \frac{\partial a_0}{\partial \xi} - \beta \frac{\partial^3 a_0}{\partial \xi^3} = 0. \tag{18}$$

This is a one-dimensional KdV equation describing the propagation of a stable planar solitary wave

$$a_0 = -A \operatorname{sech}^2 \left\{ \left(\frac{\alpha A}{12\beta}\right)^{1/2} \left(\xi + \frac{\alpha A}{3} \tau \right) \right\}. \tag{19}$$

Now we turn to discuss the solution of a ring dark soliton with azimuthal effect, i.e., solution of equation (15). An exact solution of equation (15) can be obtained by using a suitable variable transformation. In equation (15), the two terms with variable coefficient, i.e., $\frac{1}{2\tau} a_0$ and $\frac{1}{2c} \frac{1}{\tau^2} \frac{\partial^2 a_0}{\partial \eta^2}$, can be cancelled if we assume [11, 12]

$$\zeta = \xi - \frac{c}{2} \eta^2 \tau, \qquad a_0 = a_0(\zeta, \tau). \tag{20}$$

According to this transformation, we have $\frac{\partial}{\partial \xi} = \frac{\partial}{\partial \zeta}$, $\frac{\partial}{\partial \eta} = -c\eta\tau \frac{\partial}{\partial \zeta}$ and $\frac{\partial}{\partial \tau} = \frac{\partial}{\partial \tau} - \frac{c}{2}\eta^2 \frac{\partial}{\partial \zeta}$. Then equation (15) is reduced to the standard KdV

$$\frac{\partial a_0}{\partial \tau} + \alpha a_0 \frac{\partial a_0}{\partial \zeta} - \beta \frac{\partial^3 a_0}{\partial \zeta^3} = 0. \tag{21}$$

It is clear from equations (18) and (19) that equation (21) has a travelling wave solution of the form

$$a_0 = -A \operatorname{sech}^2 \left\{ \left(\frac{\alpha A}{12\beta}\right)^{1/2} \left(\zeta + \frac{\alpha A}{3} \tau \right) \right\}. \tag{22}$$

Therefore, we obtain an exact ring dark solitary wave solution of equation (15) in the following form:

$$a_0 = -A \operatorname{sech}^2 \left\{ \left(\frac{\alpha A}{12\beta}\right)^{1/2} \left[\xi + \left(\frac{\alpha A}{3} - \frac{c}{2}\eta^2 \right) \tau \right] \right\}. \tag{23}$$

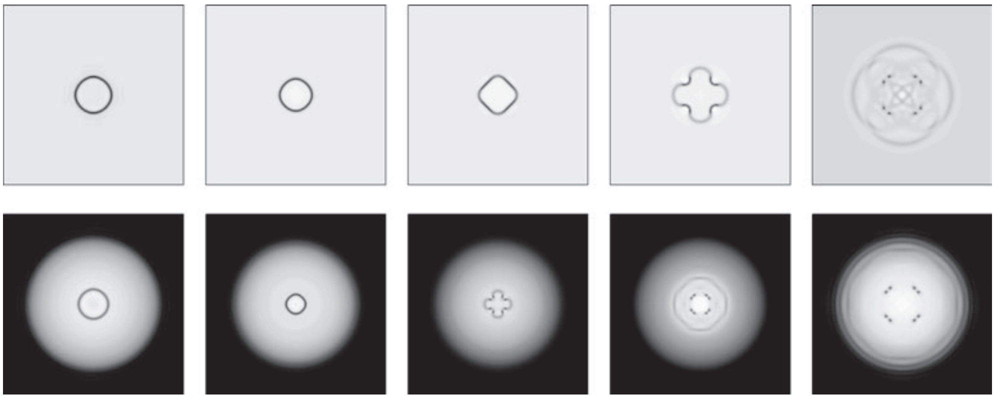

Figure 1. Evolution of ring dark soliton with $\cos\varphi(0) = 0.76$. Top row for $\omega_r/\omega_z = 0$, bottom row for $\omega_r/\omega_z = 0, 028$. From left to right: $t = 30, 40, 50, 60, 80$.

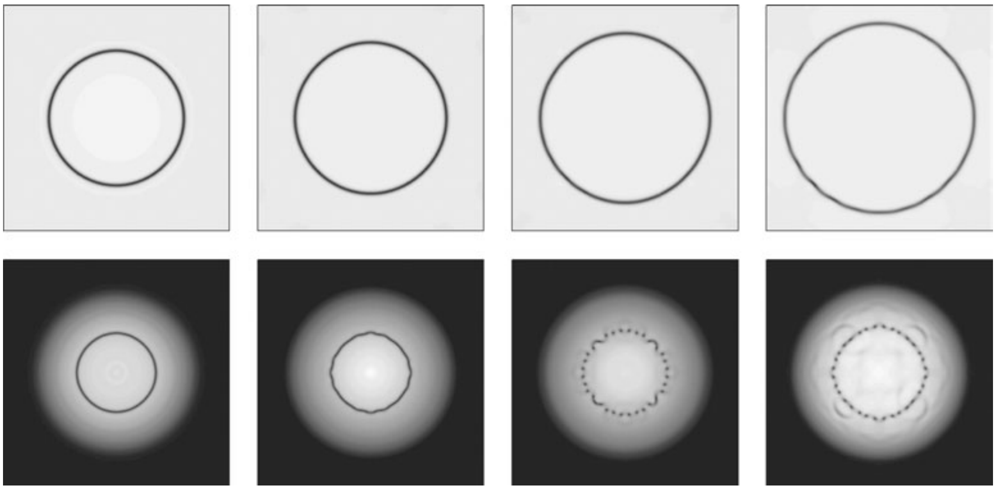

Figure 2. Evolution of ring dark soliton with $\cos\varphi(0) = 1$. Top row for $\omega_r/\omega_z = 0$, bottom row for $\omega_r/\omega_z = 0, 028$. From left to right: $t = 50, 60, 70, 80$.

It is clear that the amplitude and wave velocity of a ring dark soliton described by equation (23) are uniquely determined by the parameters of the system and only depend on the initial conditions. However, we can see from equation (23) that the phase velocity of the ring dark soliton is angle dependent in the phase. This means that the ring dark solitary wave described by cylindrical KP equation (15) will be deformed and the snake instability will occur as time goes on. On the other hand, the cylindrical KP equation (15) describes the dynamics of the ring dark solitary wave under the azimuthal effect.

Now, we investigate the stability of a ring dark soliton numerically with the effect of trapping potential in the (x, y) plane. For convenience, we rescale the wavefunction Φ in equation (4) by $\Phi = \phi/\sqrt{Q'}$, then, equation (4) reduces to

$$\mathrm{i}\frac{\partial\phi}{\partial t} = \left[-\frac{1}{2}\nabla^2 + V(r) + |\phi|^2\right]\phi. \tag{24}$$

Equation (24) is integrated numerically by means of the fourth-order Runge–Kutta scheme in time along with a second-order finite difference discretization in space. The spatial discretization step used in the simulations is typically $(\Delta x, \Delta y) = (0.2, 0.2)$. The time step of the integrator is $\Delta t = 0.0025$. The initial condition used to integrate equation (24) is [1, 8]

$$\phi(r,0) = [1 - (\omega_r/\omega_z)^2 r^2/4][\cos\varphi(0)\tanh Z(r) + \mathrm{i}\sin\varphi(0)],$$

where $Z(r) = (r - R_0)\cos\varphi(0)$, the initial ring radius $R_0 = 28.9$. The depth of the input ring soliton $\cos\varphi(0)$ and the parameter ω_r/ω_z are varied in simulations. The developments of the instability of a ring dark soliton for two cases with $\cos\varphi(0) = 0.76$ (shallow ring soliton) and $\cos\varphi(0) = 1$ (deep ring soliton) are shown in figures 1 and 2, respectively. For each case, the evolution of the ring dark soliton with the trapping potential ($\omega_r/\omega_z = 0.028$) and without the trapping potential ($\omega_r/\omega_z = 0$) are also considered. It is clear that, as time goes on, the azimuthal perturbation is developed, the ring dark soliton is deformed and the snake instability sets in finally for all cases. It is important to note that the ring dark soliton with trapping is more unstable to transverse perturbation, i.e., the trapping potential enhances the snaking instability. On the other hand, the development of the snake instability of the ring dark soliton without the trapping is postponed. We can also find from figures 1 and 2 that the expanding (for $\cos\varphi(0) = 0.76$, $\sin\varphi(0) = -0.65 < 0$) or contracting (for $\cos\varphi(0) = 1$, $\sin\varphi(0) = 0$) rate of a ring dark soliton with trapping is slower than that without the trapping.

In summary, the evolution of a ring dark soliton in a BEC with thin disc-shaped potential is studied by both perturbation method and numerical method. Theoretical analysis shows that the ring dark soliton in the BEC with azimuthal effect is governed by a cylindrical KP equation, while the ring dark soliton in the BEC without azimuthal effect is governed by a cylindrical KdV equation. The reduction to the cylindrical KP or KdV equation may be useful to understand the dynamics of the ring dark soliton and will help to get a deeper insight into the physics of the ring dark soliton.

Acknowledgment

This work is partly supported by the National Natural Science Foundation of China under grant nos 10347006 and 10475066.

References

[1] Kivshar Y S and Yang X 1994 *Phys. Rev.* E **50** R40
[2] Dreischuh A, Kamenov V and Dinev S 1996 *Appl. Phys.* B **62** 139
[3] Frantzeskakis D J and Malomed B A 1999 *Phys. Lett.* A **264** 179
[4] Nistazakis H E, Frantzeskakis D J, Malomed B A and Kevrekidis P G 2001 *Phys. Lett.* A **285** 157
[5] Neshev D *et al* 1997 *Appl. Phys.* B **64** 429
[6] Dreischuh A *et al* 2002 *Phys. Rev.* E **66** 066611
[7] Anderson B P *et al* 2001 *Phys. Rev. Lett.* **86** 2926
[8] Theocharis G *et al* 2003 *Phys. Rev. Lett.* **90** 120403
[9] Huang G, Makarov V A and Velarde M G 2003 *Phys. Rev.* A **67** 023604
[10] Perez-Garcia V M, Michinel H and Herrero H 1998 *Phys. Rev.* A **57** 3837
[11] Johnson R S 1980 *J. Fluid Mech.* **97** 701
[12] Xue J-K 2003 *Phys. Lett.* A **314** 479
[13] Weidman P D and Zakhem R 1988 *J. Fluid Mech.* **191** 557